Solution to HW 10

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§ 16.7

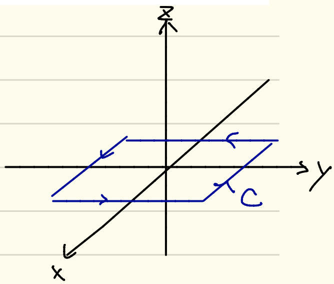
Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field **F** around the curve C in the indicated direction.

5. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

C: The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy-plane, counterclockwise when viewed from above

Sol) $\vec{F} = (y^2 + z^2)\vec{i} + (x^2+y^2)\vec{j} + (x^2+y^2)\vec{k}$.



$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix}$$

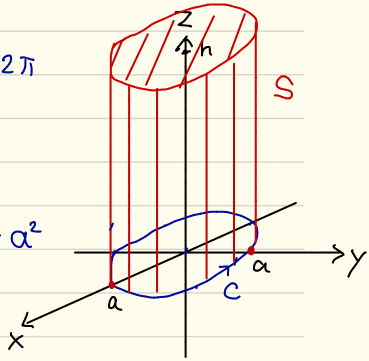
$$= (2y)\vec{i} + (2z - 2x)\vec{j} + (2x - 2y)\vec{k} \quad ; \quad \vec{n} = \vec{k}$$

$$\operatorname{curl}\vec{F}\cdot\vec{n} = 2x - 2y \quad ; \quad d\sigma = dx\,dy$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1}\int_{-1}^{1} (2x - 2y)\,dx\,dy = \int_{-1}^{1} \left[x^2 - 2xy\right]_{-1}^{1} dy$$

$$= \int_{-1}^{1} (-4y)\,dy = \left[-2y^2\right]_{-1}^{1} = 0$$

9. Let S be the cylinder $x^2 + y^2 = a^2, 0 \le z \le h$, together with its top, $x^2 + y^2 \le a^2$, $z = h$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through S.



Sol) $C : \vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}\,,\ 0 \le t < 2\pi$

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = (-a\sin t)\cdot(-a\sin t) + (a\cos t)(a\cos t) = a^2$$

$$\therefore \iint_S \nabla\times\vec{F}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r}$$

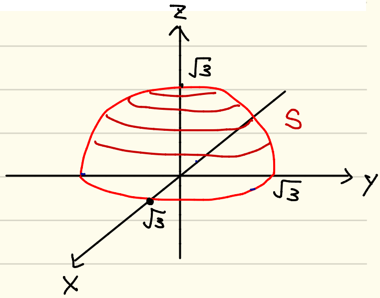
$$= \int_0^{2\pi} a^2\,dt = 2\pi a^2$$

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field **F** across the surface S in the direction of the outward unit normal **n**.

17. $\mathbf{F} = 3y\mathbf{i} + (5 - 2x)\mathbf{j} + (z^2 - 2)\mathbf{k}$

S: $\mathbf{r}(\phi, \theta) = \left(\sqrt{3}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{3}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{3}\cos\phi\right)\mathbf{k}, \quad 0 \le \phi \le \pi/2, \quad 0 \le \theta \le 2\pi$



Sol)

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix}$$

$$= -5\vec{k} \ ;$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{3}\cos\phi\cos\theta & \sqrt{3}\cos\phi\sin\theta & -\sqrt{3}\sin\phi \\ -\sqrt{3}\sin\phi\sin\theta & \sqrt{3}\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= 3\sin^2\phi\cos\theta\,\vec{i} + 3\sin^2\phi\sin\theta\,\vec{j} + 3\cos\phi\sin\phi\,\vec{k}$$

$$\therefore \iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (-15\cos\phi\sin\phi)\,d\phi\,d\theta$$

$$= 2\pi\cdot\left[\frac{-15\sin^2\phi}{2}\right]_0^{\frac{\pi}{2}} = -15\pi$$

26. Zero curl, yet field not conservative Show that the curl of

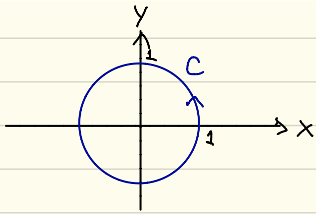
$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + z\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is not zero if C is the circle $x^2 + y^2 = 1$ in the xy-plane. (Theorem 7 does not apply here because the domain of $\mathbf{F}$ is not simply connected. The field $\mathbf{F}$ is not defined along the z-axis so there is no way to contract C to a point without leaving the domain of $\mathbf{F}$.)

Sol) $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} + z\vec{k}$.



$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix}$$

$$= (0)\,\vec{i} + (0)\,\vec{j} + \left(\frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} - \left(\frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2}\right)\right)\vec{k} = 0$$

While $C : \vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$, $0 \le t < 2\pi$

$$\frac{d\vec{r}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}\right) \cdot (-\sin t) + \left(\frac{\cos t}{\cos^2 t + \sin^2 t}\right)(\cos t) + 0 \quad = 1$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 1\,dt = 2\pi \neq 0$$

§16.8

Calculating Flux Using the Divergence Theorem

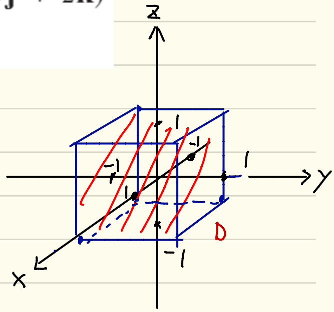
In Exercises 5–16, use the Divergence Theorem to find the outward flux of **F** across the boundary of the region D.

5. Cube $\mathbf{F} = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}$

D: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

13. Thick sphere $\mathbf{F} = \sqrt{x^2 + y^2 + z^2}\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

D: The region $1 \le x^2 + y^2 + z^2 \le 2$

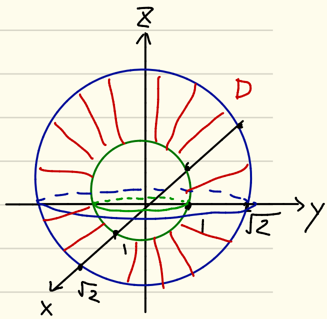


Sol) 5) $\vec{F} = (y-x)\vec{i} + (z-y)\vec{j} + (y-x)\vec{k}$

$$\nabla\cdot\vec{F} = -1 + (-1) + 0 = -2$$

$$\therefore \text{Flux} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(-2)\,dx\,dy\,dz$$

$$= 2\cdot2\cdot2\cdot(-2) = -16$$



13) $D = \{(\rho,\phi,\theta)\mid 1\le\rho\le 2,\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi\}$

$$\vec{F}(x,y,z) = \rho(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla\cdot\vec{F} = \left(x\frac{\partial\rho}{\partial x} + \rho\right) + \left(y\frac{\partial\rho}{\partial y} + \rho\right) + \left(z\frac{\partial\rho}{\partial z} + \rho\right)$$

$$= \left(\frac{x^2}{\rho} + \frac{y^2}{\rho} + \frac{z^2}{\rho}\right) + 3\rho = 4\rho$$

$$\therefore \text{Flux} = \iiint_D 4\rho\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}} 4\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\cdot\left[-\cos\phi\right]_0^{\pi}\cdot\left[\rho^4\right]_1^{\sqrt{2}} = 12\pi$$

18. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be differentiable vector fields and let a and b be arbitrary real constants. Verify the following identities.

a. $\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$

b. $\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$

c. $\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

Sol) Let $\vec{F}_1(x,y,z) = M_1(x,y,z)\vec{i} + N_1(x,y,z)\vec{j} + P_1(x,y,z)\vec{k}$

$\vec{F}_2(x,y,z) = M_2(x,y,z)\vec{i} + N_2(x,y,z)\vec{j} + P_2(x,y,z)\vec{k}$

a) LHS $= \nabla\cdot\left((aM_1+bM_2)\vec{i} + (aN_1+bN_2)\vec{j} + (aP_1+bP_2)\vec{k}\right)$

$= \left(a\frac{\partial M_1}{\partial x} + b\frac{\partial M_2}{\partial x}\right) + \left(a\frac{\partial N_1}{\partial y} + b\frac{\partial N_2}{\partial y}\right) + \left(a\frac{\partial P_1}{\partial z} + b\frac{\partial P_2}{\partial z}\right)$

$= a\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right) + b\left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right) = a(\nabla\cdot\vec{F}_1) + b(\nabla\cdot\vec{F}_2) = $ RHS

b) LHS $= \nabla\times\left((aM_1+bM_2)\vec{i} + (aN_1+bN_2)\vec{j} + (aP_1+bP_2)\vec{k}\right)$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (aM_1+bM_2) & (aN_1+bN_2) & (aP_1+bP_2) \end{vmatrix} = \left(\left(a\frac{\partial P_1}{\partial y} + b\frac{\partial P_2}{\partial y}\right) - \left(a\frac{\partial N_1}{\partial z} + b\frac{\partial N_2}{\partial z}\right)\right)\vec{i}$

$+\left(-\left(a\frac{\partial P_1}{\partial x} + b\frac{\partial P_2}{\partial x}\right) + \left(a\frac{\partial M_1}{\partial z} + b\frac{\partial M_2}{\partial z}\right)\right)\vec{j} + \left(\left(a\frac{\partial N_1}{\partial x} + b\frac{\partial N_2}{\partial x}\right) - \left(a\frac{\partial M_1}{\partial y} + b\frac{\partial M_2}{\partial y}\right)\right)\vec{k}$

$= a\left[\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right)\vec{i} + \left(-\frac{\partial P_1}{\partial x} + \frac{\partial M_1}{\partial z}\right)\vec{j} + \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right)\vec{k}\right]$

$+ b\left[\left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z}\right)\vec{i} + \left(-\frac{\partial P_2}{\partial x} + \frac{\partial M_2}{\partial z}\right)\vec{j} + \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)\vec{k}\right]$

$= a\cdot(\nabla\times\vec{F}_1) + b\cdot(\nabla\times\vec{F}_2) = $ RHS

c) $LHS = \nabla \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ M_1 & N_1 & P_1 \\ M_2 & N_2 & P_2 \end{vmatrix}$

$= \nabla \cdot \left[(N_1P_2 - N_2P_1)\vec{i} + (-M_1P_2 + M_2P_1)\vec{j} + (M_1N_2 - M_2N_1)\vec{k} \right]$

$= \frac{\partial}{\partial x}(N_1P_2 - N_2P_1) + \frac{\partial}{\partial y}(-M_1P_2 + M_2P_1) + \frac{\partial}{\partial z}(M_1N_2 - M_2N_1)$

$= \left(\frac{\partial N_1}{\partial x} \cdot P_2 + N_1 \frac{\partial P_2}{\partial x} - \frac{\partial N_2}{\partial x} \cdot P_1 - N_2 \frac{\partial P_1}{\partial x} \right) + \left(-\frac{\partial M_1}{\partial y} \cdot P_2 - M_1 \frac{\partial P_2}{\partial y} + \frac{\partial M_2}{\partial y} \cdot P_1 + M_2 \frac{\partial P_1}{\partial y} \right)$

$+ \left(\frac{\partial M_1}{\partial z} \cdot N_2 + M_1 \cdot \frac{\partial N_2}{\partial z} - \frac{\partial M_2}{\partial z} \cdot N_1 - M_2 \frac{\partial N_1}{\partial z} \right)$

$= \left[M_2 \left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z} \right) + N_2 \left(-\frac{\partial P_1}{\partial x} + \frac{\partial M_1}{\partial z} \right) + P_2 \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y} \right) \right]$

$- \left[M_1 \left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z} \right) + N_1 \left(-\frac{\partial P_2}{\partial x} + \frac{\partial M_2}{\partial z} \right) + P_1 \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y} \right) \right]$

$= \vec{F_2} \cdot (\nabla \times \vec{F_1}) - \vec{F_1} \cdot (\nabla \times \vec{F_2}) = RHS$

25. Volume of a solid region Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and suppose that the surface S and region D satisfy the hypotheses of the Divergence Theorem. Show that the volume of D is given by the formula

$$\text{Volume of } D = \frac{1}{3}\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

Sol) By Divergence Theorem, $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV$

$= \iiint_D (1+1+1)\,dV = 3\iiint_D dV = 3\cdot \mathrm{Vol}(D)$

$\therefore \mathrm{Vol}(D) = \frac{1}{3}\iint_S \vec{F}\cdot\vec{n}\,d\sigma$

29. Green's first formula Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a region D that is bounded by a closed piecewise smooth surface S. Show that

$$\iint_S f\,\nabla g\cdot\mathbf{n}\,d\sigma = \iiint_D \left(f\,\nabla^2 g + \nabla f\cdot\nabla g\right)dV. \tag{9}$$

Equation (9) is **Green's first formula**. (*Hint:* Apply the Divergence Theorem to the field $\mathbf{F} = f\,\nabla g$.)

Sol) LHS $= \iint_S (f\nabla g)\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot(f\nabla g)\,dV$ (by Divergence Theorem)

$= \iiint_D \nabla\cdot\left(f\frac{\partial g}{\partial x}\vec{i} + f\frac{\partial g}{\partial y}\vec{j} + f\frac{\partial g}{\partial z}\vec{k}\right)dV$

$= \iiint_D \left(\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)\right)dV$

$= \iiint_D \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2}\right)dV$

$= \iiint_D \left[\left(f\frac{\partial^2 g}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2} + f\frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)\right]dV$

$= \iiint_D \left[f\,\nabla^2 g + \nabla f\cdot\nabla g\right]dV = $ RHS

30. Green's second formula (*Continuation of Exercise 29.*) Interchange f and g in Equation (9) to obtain a similar formula. Then subtract this formula from Equation (9) to show that

$$\iint_S (f\,\nabla g - g\nabla f)\cdot \mathbf{n}\, d\sigma = \iiint_D (f\,\nabla^2 g - g\nabla^2 f)\, dV. \qquad (10)$$

This equation is **Green's second formula**.

Sol) Applying the formula in Q29 to pair

① (f,g): $\iint_S (f\nabla g)\cdot\vec{n}\,d\sigma = \iiint_D [f\,\nabla^2 g + \nabla f\cdot\nabla g]\,dV$

② (g,f): $\iint_S (g\nabla f)\cdot\vec{n}\,d\sigma = \iiint_D [g\,\nabla^2 f + \nabla g\cdot\nabla f]\,dV$

Consider ①−②: $\text{LHS} = \iint_S (f\nabla g)\cdot\vec{n}\,d\sigma - \iint_S (g\nabla f)\cdot\vec{n}\,d\sigma$

$= \iiint_D [f\,\nabla^2 g + \nabla f\cdot\nabla g]\,dV - \iiint_D [g\,\nabla^2 f + \nabla g\cdot\nabla f]\,dV$

$= \iiint_D [f\,\nabla^2 g - g\,\nabla^2 f]\,dV$ (since $\nabla f\cdot\nabla g = \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} = \nabla g\cdot\nabla f$)

$= \text{RHS}$